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NOTE ON A THEOREM ON ENVELOPES.

BY W. R. LONGLEY.

If a one-parameter family of plane curves $F(x, y, \alpha) = 0$ has an envelope, its rectangular equation is obtained by eliminating α from the two equations $F = 0$ and $F_\alpha = 0$.* Or a parametric representation is obtained by solving the two equations for x and y in terms of α. In order to establish conditions under which this solution is possible appeal must be made to the theory of implicit functions. In a paper in these *Annals* (second series, vol. 12, Jan., 1911, pp. 73–102) Risley and MacDonald have applied standard theorems on implicit functions to determine conditions for the existence of an envelope. In the first part of the discussion it is assumed that the curves are given in the explicit form $y = f(x, \alpha)$, and the authors have stated (p. 86) a "fundamental theorem" which is said to "summarize the facts concerning envelopes of the family of curves given in the explicit form $y = f(x, \alpha)$." This theorem is the following:

"Given a one-parameter family of curves $y = f(x, \alpha)$, where $f(x, \alpha)$ is an analytic function of x and α in the neighborhood of (x_0, α_0), such that

$$(1) \qquad f_{\alpha, x^i}(x_0, \alpha) \equiv 0, \qquad i = 0, 1, 2, \cdots, m - 1,$$

but

$$f_{\alpha, x^m}(x_0, \alpha) \not\equiv 0,$$

$$(2) \qquad f_{\alpha^j}(x, \alpha_0) \equiv 0, \qquad j = 1, 2, \cdots, n,$$

but

$$f_{\alpha^{n+1}}(x, \alpha_0) \not\equiv 0;$$

then, if

$$(3) \qquad f_{\alpha^{n+1}, x^m}(x_0, \alpha_0) \neq 0,$$

the curves $y = f(x, \alpha)$ have no envelope in the neighborhood of (x_0, y_0), except a point envelope which occurs whenever $m \geqq 1$; but, if

$$(3') \qquad f_{\alpha^{n+1}, x^m}(x_0, \alpha_0) = 0,$$

the family $y = f(x, \alpha)$ has an envelope composed of one or more curves through the point (x_0, y_0); also, whenever $m \geqq 1$, a point envelope at that point.

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* F_α indicates the derivative of F with respect to α. This notation is used throughout the paper.

"When $m = 0$, the notation $i = 0, 1, 2, \cdots, m - 1$ is meaningless and (1) must be replaced by $f_\alpha(x_0, \alpha) \not\equiv 0$."

The form of statement is designed particularly to cover the case when $y_0 \equiv f(x_0, \alpha)$ is an identity in α. Otherwise, as explained in the last sentence of the theorem, the notation of condition (1) has no meaning. When $m \geqq 1$, the theorem as stated is not established by the argument. The hypothesis places certain conditions on the character of the function f in the neighborhood of the " point " (x_0, α_0). The theorem states a conclusion for the neighborhood of the point $P_0(x_0, y_0)$. When $m \geqq 1$, the " point " (x_0, α_0) of the function f is not determined by the geometric point (x_0, y_0). There is no mention of this fact in the proof.

Since it is the " point " (x_0, α_0) that is used throughout the argument, the conclusion applies not to the point (x_0, y_0) but to the curve $y = f(x, \alpha_0)$ at the point (x_0, y_0). The content of the theorem actually proved is: If the hypotheses (1), (2), and (3) are satisfied, the curve $y = f(x, \alpha_0)$ does not touch a branch of the envelope at (x_0, y_0) and no curve of the system for values of α sufficiently near to α_0 touches a branch of the envelope in the neighborhood of (x_0, y_0), except a point envelope which occurs whenever $m \geqq 1$; but if (3′) is satisfied, the family $y = f(x, \alpha)$ has an envelope composed of one or more branches which are tangent at the point (x_0, y_0) to the curve $y = f(x, \alpha_0)$; also, whenever $m \geqq 1$, a point envelope at that point.

If we attempt to obtain the facts for the neighborhood of P_0 the difficulties encountered are considerable. For the ordinary geometric interpretation suppose we consider only real values of x, y, and α. In order to draw the negative conclusion (3) of the theorem quoted it would be necessary to have the hypotheses fulfilled for $x = x_0$ and every real* value of α. It is easy to show by simple examples that if there is one real value of α for which the hypotheses are not satisfied, it is possible to have an envelope which comes within every arbitrarily small neighborhood of P_0. For this purpose consider the system of curves

$$y = f(x, \alpha) = 2\alpha x^2 - \alpha^2 x^3,$$

and inquire about the nature of the envelope near the origin. Here $m = 2$, $n = 0$ and the hypotheses of the theorem are satisfied for every real finite value of α. To examine the value $\alpha = \infty$ we write $\alpha = 1/\beta$ and set $\beta = 0$. Then the hypotheses are not satisfied because the function f is not analytic. The envelope of the system (in addition to the point envelope at the origin) is the straight line $y = x$, which is given in parametric form by the equations $x = 1/\alpha$, $y = 1/\alpha$. For every finite value $\alpha = \alpha_0$ the curve $y = 2\alpha_0 x^2 - \alpha_0^2 x^3$ is tangent to the envelope at the point $x = 1/\alpha_0$, $y = 1/\alpha_0$. By taking α_0 large enough we see that the envelope comes within every arbitrarily small neighborhood of the origin.

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* This includes $\alpha = \infty$, which would be covered by examining the curve $y = f(x, 1/\beta)$ for $\beta = 0$.

A particularly interesting example in this connection is the following:

$$y = \alpha \sin \frac{x}{\alpha}.$$

Every curve of the system passes through the origin, and at this point the hypotheses of the theorem ($m = 1$, $n = 0$) are satisfied for every real value of α except $\alpha = 0$, $\alpha = \infty$. Every curve of the system is tangent at the origin to the line $y = x$. The envelope consists of the infinite number of straight lines given in parametric form by the equations

$$x = \alpha k_i, \qquad y = \alpha \sin k_i, \qquad\qquad i = 1, 2, \cdots,$$

where k_i denote the roots of the equation

$$k_i = \tan k_i.$$

Every curve of the system touches each branch of the envelope twice (once for x positive and once for x negative) and, with the exception of the origin, every branch of the envelope is touched at every one of its points by one curve of the system. In this example we have in every arbitrarily small neighborhood of the origin an infinite number of branches of the envelope, not one of which can be detected by the theorem.

The known theorems on implicit functions demand that the functions and certain derivatives shall be finite for the particular values of the arguments involved. Hence the examples given above are sufficient to show that it is impossible to derive exhaustive criteria for the nature of the envelope in the neighborhood of a point by using the implicit function theorems and the values of f and its derivatives at (x_0, y_0).

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